The Sitting Closer to Friends than Enemies Problem in the Circumference

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Abstract The Sitting Closer to Friends than Enemies (SCFE) problem is to find an embedding in a metric space for the vertices of a given signed graph so that, for every pair of incident edges with different sign, the positive edge is shorter (in the metric of the space) than the negative edge. In this document, we present new results regarding the SCFE problem when the metric space in consideration is the circumference. Our main results say that, given a signed graph, it is NP-complete to decide whether such an embedding exists in the circumference or not. Nevertheless, if the given signed graph is complete, then such decision can be made in polynomial time. In particular, we prove that, given a complete signed graph, it has such an embedding if and only if its positive part is a proper circular arc graph.

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1 Introduction

Consider a group of people that may have positive or negative interactions between them. For instance, they may be friends, or enemies, or they may not know each other. Now, sit them all in a big circular table so that each person is first surrounded by their friends and then, farther than all their friends, the person can see their enemies. The problem of finding such a placement in a circular table is known as the Sitting Closer to Friends than Enemies (SCFE) problem in the circumference.

A reasonable way to represent the group of people is using a signed graph, i. e., a graph with a sign assignment on its edges. Therefore, if each vertex of the graph represents one person, then, two friends are connected with a positive edge, two enemies are connected with a negative edge, and two unknown people aren't connected in the graph. In this document, we show that the SCFE problem is NP-Complete, but when the signed graph is complete (all its edges are present) the problem can be solved in polynomial time. Furthermore, we give a characterization of the set of complete signed graphs for which the problem has a solution.

2 Definitions and Notation

In this document, we consider signed graphs that are finite, undirected, connected, loopless and without parallel edges. For a signed graph $G = (V, E^+ \cup E^-)$, we denote by $V(G)$, $E^+(G)$ and $E^-(G)$ the set of vertices, positive edges and negative edges, respectively. When the signed graph under consideration is clear, we use only V, E^+ and E^-. It is worth noting that in every signed graph it holds $E^+ \cap E^- = \emptyset$. The number of vertices of a signed graph G is denoted by $n := |V(G)|$, the number of positive edges is denoted by $m^+ := |E^+|$, and the number of total edges of a graph (positive plus negative edges) is denoted by $m := |E^+(G)| + |E^-(G)|$.

The positive or negative edge $\{i, j\}$ is denoted ij. If $ij \in E^+$ (resp., $ij \in E^-$), we say that i is a *positive neighbor* (resp., *negative neighbor*) of j and vice versa. The set of positive (resp., negative) neighbors of i is denoted $N^+(i)$ (resp., $N^-(i)$). Additionally, the *closed positive neighborhood* of i is defined as $N^+[i] := N^+(i) \cup \{i\}$.

The signed graph H is a *signed subgraph* of the signed graph G if and only if:

$$V(H) \subseteq V(G) \quad \wedge \quad E^+(H) \subseteq E^+(G) \quad \wedge \quad E^-(H) \subseteq E^-(G).$$

The subgraph $H = (V(G), E^+(G) \cup \emptyset)$ of G that contains all the vertices of G, all the positive edges and none of the negative edges of G is called the *positive subgraph* of G and it is denoted by G^+. Even though, the positive subgraph of a signed graph is a signed graph, it can also be seen as a graph (with no signs on its edges) since all its edges have the same sign. A signed graph is said to be *complete* if and only if for every pair i and $j \in V$ such that $i \neq j$, $ij \in E^+$ or $ij \in E^-$.

The *circumference*, w.l.o.g.[1], is defined as the set $\mathcal{C} = \{(x,y) : \sqrt{x^2 + y^2} = 1\} \subseteq \mathbb{R}^2$. Every point p in $\mathcal{C}$ is fully determined by the angle, in $[0, 2\pi[$, formed by the points $(0,1)$, the origin $\vartheta = (0,0)$ and the point p. Hence, as an abuse of notation, we use that angle in $[0, 2\pi[$ to denote $p \in \mathcal{C}$. For example, the point $(0,1)$ is the point 0. The *distance* between two pints p and q in $\mathcal{C}$ is the measure of the smallest angle formed by p, ϑ, and q. Hence, if $q \leq p$, the distance between p and q is defined as:

$$d(q,p) = d(p,q) := \min\{(p-q) \bmod 2\pi, (2\pi - p + q) \bmod 2\pi\}.$$

The pair $(\mathcal{C}, d)$ is the metric space that we consider in this work.

A *drawing* of a signed graph G in $\mathcal{C}$ is an injection $D : V \to \mathcal{C}$ of the vertex set V into $\mathcal{C}$. A drawing D of G in $\mathcal{C}$ is said to be *valid* if:

$$\forall i \in V, \forall j \in N^+(i), \forall k \in N^-(i) \text{ it holds } d(D(i), D(j)) < d(D(i), D(k)). \quad (1)$$

In the case that there exists a valid drawing of a given signed graph G in $\mathcal{C}$, we say that *G has a valid drawing* in $\mathcal{C}$. Otherwise, we simply say that G is a signed graph without valid drawing in $\mathcal{C}$. The definition of a valid drawing captures the requirement that every vertex is placed closer to its friends than to its enemies in a valid drawing. Hence, the Sitting Closer to Friends than Enemies problem in the circumference is:

Definition 1 *(SCFE problem)* Given a signed graph G decide whether G has a valid drawing in $\mathcal{C}$.

If, for a given signed graph G, the SCFE problem has a positive answer, then, we are also interested in finding a valid drawing for G in $\mathcal{C}$.

Before we continue with the presentation of our contributions, we give some more notation and definitions. Given $p \in \mathcal{C}$, we define the *right half* and *left half* of p in $\mathcal{C}$ as $M_r(p) := \{(p+t) \bmod 2\pi : 0 \leq t \leq \pi\}$ and $M_l(p) := \{(p-t) \bmod 2\pi : 0 \leq t \leq \pi\}$, respectively. Therefore, given two points $p, q \in \mathcal{C}$, it holds that $p \in M_r(q) \iff q \in M_l(p)$.

Finally, a drawing of a signed graph in $\mathcal{C}$ induces a cyclic order of its vertices. Given a signed graph G and a drawing D of G in $\mathcal{C}$, we say that i is smaller than j according to D if, starting from the point 0 and traveling $\mathcal{C}$ in clockwise direction, we first find i and then j. In such case, we denote $i <_D j$ or simply $i < j$ if the drawing D is clear by the context. Now, given a signed graph G and a drawing D of G in $\mathcal{C}$, we relabel the vertices naming 1 the first vertex in the ordering induced by D, 2 the second vertex, and so on until n the last vertex. Hence, it holds $1 <_D 2 <_D 3 <_D \cdots <_D n$. It is worth noting that it also holds $D(1) < D(2) < \cdots < D(n)$, when, again, $D(i)$, as an abuse of notation, is the angle in $[0, 2\pi[$ formed by the point 0, the origin ϑ and of the point in which D injects vertex i. Even though, the order as defined above is not cyclic, we provide it with a cyclic structure due to the circular characteristic of the space. Hence, the vertex set $V = \{1, 2, 3, \ldots, n\}$ is considered cyclically ordered set where $1 < 2 < 3 < \ldots < n < 1$.

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[1] It is worth noting that this definition can be modified changing the radius and/or the center of the circumference and all the results presented in this document will remain valid.

3 Related Work and Our Contributions

The SCFE problem was first introduced by Kermarrec et al. in [7]. In their work, Kermarrec and Thraves focused in the SCFE problem on $\mathbb{R}$. They presented some families of signed graphs without valid drawing on $\mathbb{R}$. They gave a characterization of the set of signed graphs with valid drawing on $\mathbb{R}$. Such characterization was also used to give a polynomial time algorithm to decide whether a complete signed graph has a valid drawing or not, and to find one in case the answer is positive.

Afterwords, Cygan et al. in [2] proved that the SCFE problem on $\mathbb{R}$ is NP-Complete. Nevertheless, they gave a new and more precise characterization of the set of complete signed graphs with valid drawing on $\mathbb{R}$ that says that a complete signed graph G has a valid drawing on $\mathbb{R}$ if and only if G^+ is a unit interval graph.

Based on the previous NP-Completeness result, Garcia Pardo et al. in [9] studied an optimization version of the SCFE problem on $\mathbb{R}$ where the goal is to find a drawing that minimizes the number of violations of condition (1) in Definition 1. They proved that when the signed graph G is complete, local minimums for their optimization problem coincide with local minimums of the well known *Quadratic Assignment* problem applied to G^+. Moreover, they studied experimentally two heuristics, showing that a greedy heuristic has a good performance at the moment of recognition of graphs that have an optimal solution with zero errors (problem that is NP-Complete).

Spaen et al. in [10] studied the SCFE problem form a different perspective. They studied the problem of finding $L(n)$, the smallest dimension k such that any signed graph on n vertices has a valid drawing in $\mathbb{R}^k$, with respect to euclidean distance. They showed that $L(n) = n - 2$ by demonstrating that any signed graph on n vertices has a valid drawing in $\mathbb{R}^{n-2}$ and that there exists a signed graph on n vertices that does not have a valid drawing in $\mathbb{R}^{n-3}$

Finally, Becerra in [1] studied the SCFE problem in trees. The main result of her work was to prove that a complete signed graph G has a valid drawing in a tree if and only if G^+ is strongly chordal.

Our Contributions: In this work we study the SCFE problem in the circumference $\mathcal{C}$. The main result of this work says in Theorem 1 that a signed graph G has a valid drawing in $\mathcal{C}$ if and only if G has a way to be completed so that the positive part of the completed version of G is a proper circular arc graph. From this characterization, three interesting corollaries are concluded. First we conclude in Corollary 1 that the SCFE problem in the circumference is NP-Complete. Nevertheless, we also conclude in Corollary 2 that, the SCFE problem in $\mathcal{C}$ restricted to complete signed graph becomes polynomial. Finally, we conclude in Corollary 3 that, the complexity of the SCFE problem in the circumference can be parametrized with respect to the number of missing edges in a signed graph. Indeed, if we consider that the number of missing edges in a signed graph k is a constant, then the SCFE problem becomes polynomial (but exponential in k).

Structure of the Document: The rest of the document has the following structure. In Section 4, we present two instrumental lemmas that we use in Section 5 to prove our main results. Finally, in Section 6, we present our conclusions.

4 Instrumental Lemmas

In this section, we introduce two lemmas that we use later to prove the main result of this document. The first lemma is about proper circular arc graphs, and will help to relate them with valid drawings. The second lemma tells how to go from a relaxation of condition (1) in Definition 1 to a valid drawing.

Circular Arc Graphs: A *circular arc* graph is the intersection graph of a set of arcs on the circumference. It has one vertex for each arc in the set, and an edge between every pair of vertices corresponding to arcs that intersect. The set of arcs that corresponds to a graph G is called the *circular arc model* of G. A *proper circular arc* graph is a circular arc graph for which there exists a corresponding circular arc model such that no arc properly contains another. Such model is called *proper circular arc model*. On the other hand, a *unit circular arc* graph is a circular arc graph for which there exists a corresponding circular arc model such that each arc is of equal length. Such model is called *unit circular arc model*.

The first characterization of circular arc graphs is due to Alan Tucker in [11]. The same author in [13] also presented the first polynomial time recognition algorithm for this family, which runs in $\mathcal{O}(n^3)$ time. Ross M. McConnell in [8] presented the first $\mathcal{O}(n + m)$ time recognition algorithm for circular arc graphs. Recognition of a proper circular arc graph and construction of a proper circular arc model can both be performed in time $\mathcal{O}(n + m)$ as it was proved by Deng et al. in [3]. On the other hand, Duran et al. in [4] proved that unit circular arc graphs can be recognized in $\mathcal{O}(n^2)$ time. Later, Lin and Szwarcfiter gave an $\mathcal{O}(n)$ time recognition algorithm for unit circular arc graphs that also constructs a unit circular arc model for the graph in consideration.

From the definition, we can see that every proper circular arc graph is also a circular arc graph. On the other hand, it is known that the opposite contention does not hold. For example the *net* (see Figure **??**) is a circular arc graph that does not have a proper circular arc model. In the same line, every unit circular arc graph is also a proper circular arc graph, but, Alan Tucker in [12] gave a characterization of proper circular arc graphs which are not unit circular arc graphs. It is worth noting that Tucker's characterization uses crucially the fact that all the unit arcs are closed or all are open. Actually, the graph in Figure **??** does not have a unit circular arc model only because the arcs are restricted to be all closed or all open (as we will see later). Kaplan and Nassbaum in [6] pointed out that the family of unit circular arc graphs does not change if all the arcs are restricted to be open or closed. Nevertheless, if in a unit circular arc model arcs are allowed to be either open or closed the family changes. Indeed, every proper circular arc graph has an arc model with

arcs of the same length, where arcs are allowed to be open or closed, and this is our first instrumental lemma. We have to say that this lemma was pointed out in [6] and deduced from a construction from [12]. Nevertheless, none of these two articles stated it as a result. Hence, since we use it in later sections, we believe that it is important to state it as a lemma.

For the sake of completeness, we also point out the fact that circular arc models and proper circular arc models are not restricted to have closed or open arcs. Indeed, it is always possible to perturb the arcs in a circular arc model and in a proper circular arc model so that no two arcs share an endpoint. Hence, whether arcs are open or closed in these models does not affect the corresponding graph.

Lemma 1 *([12]) Let G be a proper circular arc graph. Then G has an arc model with arcs of the same length, where arcs can be either open or closed.*

The construction of an arc model with arcs of the same length, where arcs can be either open or closed, for a proper circular arc graph is exactly the same construction that appears in the proof of Theorem 4.3 in [12]. However, when unit arcs create new intersections in their end points and they can not be moved to avoid these new intersections, then the end point of the arc is deleted from the arc creating an open arc of unit length.

Almost Valid Drawing: Now, we introduce a less restricted version of a valid drawing. Let G be a signed graph and D be a drawing of G in $\mathcal{C}$. We say that D is *almost valid* if and only if there exists δ such that $\forall i \in V, \forall j \in N^+(i), \forall k \in N^-(i)$ it holds:

$$d(D(i), D(j)) \leq \delta \leq d(D(i), D(k)). \tag{2}$$

Restriction (1) is stronger than restriction (2). Hence, every valid drawing is as well almost valid. Nevertheless, there are almost valid drawings that are not valid. The next lemma states that it is possible to obtain a valid drawing for a given signed graph from an almost valid drawing of it.

Lemma 2 *Let G be a signed graph and D be an almost valid drawing of G in $\mathcal{C}$. Then, G has a valid drawing in $\mathcal{C}$.*

Proof Consider the almost valid drawing D of G in $\mathcal{C}$ and its corresponding cyclic labeling of V. Let us define $\widehat{V} \subseteq V$ as the set of all vertices for which restriction (1) is violated, i.e,:

$$\widehat{V} := \{i \in V : \exists j \in N^+(i) \wedge k \in N^-(i), d(D(i), D(j)) = \delta = d(D(i), D(k))\}.$$

If $\widehat{V} = \emptyset$, then D is valid.

Let us assume, then, that $\widehat{V} \neq \emptyset$. Let a be a vertex in $\widehat{V}$. We define four important vertices for a as follows:

– the *farthest friend of a on its left half of $\mathcal{C}$*

$$a_l^+ := \operatorname*{argmax}_{\{j:j \in N^+(a)\} \cap \{j:D(j) \in M_l(D(a))\}} d(D(a), D(j)),$$

– the *closest enemy of a on its left half of $\mathcal{C}$*

$$a_l^- := \underset{\{j:j\in N^-(a)\}\cap\{j:D(j)\in M_l(D(a))\}}{\mathrm{argmin}} d(D(a), D(j)),$$

– the *farthest friend of a on its right half of $\mathcal{C}$*

$$a_r^+ := \underset{\{j:j\in N^+(a)\}\cap\{j:D(j)\in M_r(D(a))\}}{\mathrm{argmax}} d(D(a), D(j)),$$

– the *closest enemy of a on its right half of $\mathcal{C}$*

$$a_r^- := \underset{\{j:j\in N^-(a)\}\cap\{j:D(j)\in M_r(D(a))\}}{\mathrm{argmin}} d(D(a), D(j)).$$

It is worth noting that the cases when $N^-(a) = \emptyset$ or $N^+(a) = \emptyset$ do not apply since in these cases $a \notin \widehat{V}$. On the other hand, we also remark the fact that $a_r^- = (a_r^+ + 1) \bmod n$ and $a_l^- = (a_l^+ - 1) \bmod n$. Additionally, since D is an injection, for any given $a \in \widehat{V}$, only one of the next situations is possible:

$$d(D(a), D(a_r^+)) = \delta = d(D(a), D(a_l^-)),$$

or,

$$d(D(a), D(a_l^+)) = \delta = d(D(a), D(a_r^-)).$$

Without loss of generality, we assume that:

$$d(D(a), D(a_r^+)) = \delta = d(D(a), D(a_l^-)).$$

If this is not the case, then we can reflect the drawing along the axis that goes through a and the center of the circumference, and, by symmetry, we obtain the desired case.

Now, we transform D into an almost valid drawing for which the set of vertices that violate restriction (1) does not contain a and does not contain any new vertex. We define $\varepsilon := \frac{1}{4} \min_{i\in V} d(D(i), D((i+1) \bmod n))$. Since D is an injection, it holds that $\varepsilon > 0$. Furthermore, due to the definition of ε it also holds that $\delta > \varepsilon$. Now, we define the following drawing $D' : V \to \mathcal{C}$,

$$D'(i) = \begin{cases} D(i) & i \neq a \\ D(i) + \varepsilon & i = a \end{cases}$$

We first observe that, since $\varepsilon < d(D(i), D((i+1) \bmod n))$, the cyclic order of V does not change. Therefore, the cyclic labeling according to D is the same as the cyclic labeling according to D'. Hence, in this new drawing the vertices a_l^+, a_r^+, a_l^- and a_r^- remain the same.

We show now that a does not violate restriction (1) anymore. Let us analyze the new distance from a to its farthest friends and closest enemies. First the farthest friends:

$$\begin{aligned} d(D'(a), D'(a_r^+)) &= d(D(a), D(a_r^+)) - \epsilon \\ &= \delta - \epsilon \\ &< \delta, \end{aligned}$$

and

$$d(D'(a), D'(a_l^+)) = d(D(a), D(a_l^+)) + \epsilon$$
$$= d(D(a), D(a_l^-)) - d(D(a_l^-), D(a_l^+)) + \epsilon$$
$$= \delta - d(D(a_l^-), D(a_l^+)) + \epsilon$$
$$< \delta,$$

where the last inequality is obtained since $d(D(a_l^-), D(a_l^+)) + \epsilon < 0$. On the other hand, if we do the same analysis for the closest enemies of a we obtain that:

$$d(D'(a), D'(a_l^-)) = d(D(a), D(a_l^-)) + \epsilon$$
$$= \delta + \epsilon$$
$$> \delta,$$

and

$$d(D'(a), D'(a_r^-)) = d(D(a), D(a_r^-)) - \epsilon$$
$$= d(D(a), D(a_r^+)) + d(D(a_r^+), D(a_r^-)) - \epsilon$$
$$= \delta + d(D(a_r^+), D(a_r^-)) - \epsilon$$
$$> \delta,$$

where the last inequality is obtain since $d(D(a_r^+), D(a_r^-)) - \epsilon > 0$. In conclusion, vertex a does not violate restriction (1) in D'.

Finally, consider $b \in V$ such that $b \neq a$. If b did not violate restriction (1) in D, then it does not do it in D' neither. Indeed, the distance between b and any other vertex different than a do not change due to the construction of D'. Regarding the distance between b and a, we can say that, if $b \in N^+(a)$ then:

$$d(D'(a), D'(b)) \leq \max\{d(D'(a), D'(a_r^+)), d(D'(a), D'(a_l^+))\} < \delta,$$

while if $b \in N^-(a)$, then:

$$d(D'(a), D'(b)) \geq \min\{d(D'(a), D'(a_r^-)), d(D'(a), D'(a_l^-))\} > \delta.$$

In conclusion, transforming D into D' has decreased the size of $\widehat{V}$ by at least one. Hence, repeating the process at most $|\widehat{V}|$ times, we obtain a valid drawing of G in $\mathcal{C}$.   □

5 Characterization of Signed Graphs with Valid Drawing in the Circumference

In this section we present the main result of this work. Given a signed graph G, a *completion* of G is a set of decisions of the type $ij \in E^+$ or $ij \in E^-$ for all $ij \notin E^+ \cup E^-$. Given a signed graph G, we denote the complete signed graph obtained by a completion of G by C_G and its positive subgraph by C_G^+

Theorem 1 *Let G be a signed graph. Then G has a valid drawing in $\mathcal{C}$ if and only if there exists C_G, a completion of G, such that C_G^+ is a proper circular arc graph.*

Proof Let us first point out the fact that, if a signed graph G has a valid drawing in $\mathcal{C}$, then any signed subgraph H of G also has a valid drawing in $\mathcal{C}$. This affirmation can be obtained by simply considering exactly the same valid drawing D for G but now for H. Indeed, all the restrictions 1 that need to be satisfied by a valid drawing for H are satisfied already by D. Moreover, by deleting edges or vertices from G to obtain H, the only thing that may happen is that some of the restrictions 1 that are satisfied by D are not required by H, but this does not harm the validity of D for H.

Consider now a signed graph G. Let us first assume that there exists a completion of G such that C_G^+ is a proper circular arc graph. We construct a valid drawing for C_G^+ in $\mathcal{C}$. Hence, due to the affirmation of the previous paragraph, this valid drawing is also a valid drawing for G (since G is a signed subgraph of C_G^+).

Lemma 1 says that C_G^+ has an arc model with arcs of the same length, let say δ. Let us denote by s_i the clockwise end of the arc corresponding to vertex i in this model. We define now the $D : V \to \mathcal{C}$ by $D(i) := s_i$. The drawing D satisfies $d(D(i), D(j)) \leq \delta$ for all $ij \in E^+$ and $d(D(i), D(j)) \geq \delta$ for all $ij \in E^-$. Hence, D is an almost valid drawing. Therefore, by Lemma 2 D can be transformed into a valid drawing for C_G^+ in $\mathcal{C}$.

In the opposite direction, let G be a signed graph with a valid drawing D in $\mathcal{C}$. We will show that there exists a completion of G such that C_G^+ is a proper circular arc graph. For the purpose of this proof, we consider every vertex to be a positive neighbor of itself. Consider $V = \{1, 2, 3 \ldots, n\}$ ordered cyclically according D. For every vertex $i \in V$, we use the definition and notation of i_l^+, the farthest friend of i on its left half of $\mathcal{C}$, and i_r^+ the farthest friend of i on its right half of $\mathcal{C}$, as in the previous section.

The completion of G adds positive edges between every vertex i and every vertex j such that $i < j < i_r^+$ or $i_l^+ < j < i$, when $ij \notin E^+ \cup E^-$. The rest of the pairs $ij \notin E^+ \cup E^-$ are set to be negative edges. We call C_G^+ the positive subgraph of this completion of G. We shall show that C_G^+ is a proper circular arc graph.

For each $i \in V$ we consider the arc A_i that goes between the middle point between i_l^+ and i until the middle point between i and i_r^+. That is, A_i is defined as follows:

$$A_i = \left[\left(\frac{D(i) + D(i_l^+)}{2} \right) \bmod 2\pi, \left(\frac{(D(i) + D(i_r^+))}{2} \right) \bmod 2\pi \right].$$

We claim that $\{A_i, i \in V\}$ is a proper circular arc model of C_G^+.

Let us consider two vertices $i, j \in V$. Note that j belongs either to the left half or to the right half of i in $\mathcal{C}$. Let us assume that j belongs to right half of f i in $\mathcal{C}$, otherwise we can change the roles of i and j in the proof. Without loss of generality, we can assume that $D(i) = 0$. Then $D(j) \in (0, \pi]$.

Consider first the case when $ij \in E^-$. In order to prove that the arcs A_i and A_j have empty intersection, we have to show that A_i does not cross A_j neither at left nor at the right hand side of A_j. Since j is in $N^-(i)$, it holds that $D(j_r^+) < 2\pi$, and then:

$$\frac{D(j) + D(j_r^+)}{2} < \frac{\pi + 2\pi}{2} = \frac{3\pi}{2}.$$

On the other hand, either $D(i_l^+) = 0$ or $D(i_l^+) \geq \pi$. Then, it holds that:

$$\text{either } \frac{D(i) + D(i_l^+)}{2} = 0 \text{ or } \frac{D(i) + D(i_l^+)}{2} \geq \frac{3\pi}{2}.$$

In any case, the right part of A_i does not overlap the left side of A_j.

Now, since D is a valid drawing, we have that

$$D(i_r^+) < D(j) \text{ and } D(i) < D(j_l^+),$$

which allows us to conclude that:

$$\frac{D(i) + D(i_r^+)}{2} < \frac{D(i) + D(j)}{2} < \frac{D(j_l^+) + D(j)}{2},$$

and therefore the right extreme of A_i does not intersect the left extreme of A_j. We conclude that when $ij \in E^-$, $A_i \cap A_j = \emptyset$.

Let us now consider the case when $ij \in E^+$. Let us consider for this case that $D(j_l^+) = 0$. Since D is a valid drawing, we have that:

$$D(j_l^+) \leq D(i) \text{ and } D(j) \leq D(i_r^+),$$

therefore,

$$\frac{D(j) + D(j_l^+)}{2} \leq \frac{D(j) + D(i)}{2} < \frac{D(j) + D(j)}{2} = D(j).$$

It follows that $\frac{D(j)+D(i)}{2} \in A_j$. Similarly we have:

$$D(i) = \frac{D(i) + D(i)}{2} < \frac{D(j) + D(i)}{2} \leq \frac{D(i_r^+) + D(i)}{2},$$

and then $\frac{D(j)+D(i)}{2} \in A_i$. We conclude that $A_i \cap A_j \neq \emptyset$.

To conclude the proof, we need to show that no arc properly contains another one. Notice that we only need to consider the case of two vertices i and j such that $A_i \cap A_j \neq \emptyset$. Then, by the previous analysis, we have that $ij \in E^+$. We assume that $D(i) = 0$ and $D(j) \in (0, \pi]$. Let $x = \frac{D(j)+D(i_r^+)}{2}$. First, observe that $x \notin A_i$, since $\frac{D(i)+D(i_r^+)}{2} < x \leq \pi$. Also, $d(j, i_r^+) \leq d(i, i_r^+)$ and then $ji_r^+ \in E^+$. Then $D(j) \leq D(i_r^+) \leq D(j_r^+)$ and $D(j) \leq x \leq \frac{D(j)+f(j_r^+)}{2}$. This shows that $x \in A_j$. A symmetric argument leads to find $y = \frac{D(i)+D(j_l^+)}{2} \in A_i \setminus A_j$, which ends the proof. $\square$

The *graph sandwich* problem for a given property is to decide whether, given two graphs on the same set of vertices $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, there exists a graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$ which satisfies the given property. From Theorem 1, using the graph sandwich problem for proper circular arc graphs, we can conclude the NP-Completness of the SCFE problem. Indeed, in our case, given a signed graph G, deciding if there exists a completion of G such that C_G^+ is a proper circular arc graph is equivalent to solve the graph sandwich problem for the proper circular arc property with $G_1 = (V, E^+)$ and $G_2 = (V, E^+ \cup \{ij : ij \notin E^+ \cup E^-\})$ Since Golumbic et al. in [5] proved that the graph sandwich problem for proper circular arc graphs is NP-Complete, we conclude the following corollary.

Corollary 1 *The SCFE problem is NP-Complete.*

On the positive side, if the signed graph is complete, there is no completion to search. Hence, Theorem 1 in that case says: *a complete signed graph G has a valid drawing in $\mathcal{C}$ if and only if G^+ is a proper circular arc graph.* Since, recognition of proper circular arc graphs can be done in $\mathcal{O}(n+m)$ time. Therefore, we can conclude the following corollary.

Corollary 2 *Let G be a complete signed graph. Then deciding whether G has a valid drawing in $\mathcal{C}$ can be done in $\mathcal{O}(n + m^+)$ time.*

Finally, if G is not complete but the number of missing edges does not depend on n, there is a constant number of completions for G. Hence, an exhaustive search on all the completions of G, and repeatedly deciding if C_G^+ is a proper circular arc graph, tells us if there exists a completion that does the trick. Therefore, in the worst case, in time $\mathcal{O}(2^k(m + n))$, where k is the number of missing edges, we can know if G has a valid drawing in $\mathcal{C}$. Hence, the following corollary holds.

Corollary 3 *Let G be a signed graph such that it has k missing edges (where k does not depend on n). Then deciding whether G has a valid drawing in $\mathcal{C}$ can be done in $\mathcal{O}(2^k n^2)$ time (polynomial in n).*

6 Conclusions

In conclusion, we can say that if you have to solve the SCFE problem in the circumference for a graph that is not complete, then the problem will be hard and the complexity will depend exponentially on the number of edges missing in the signed graph. If you are willing to pay that cost, an exhaustive search to find if the signed graph has a completion so that its positive part is a proper circular arc graph shall work.

If the input signed graph is complete, first you have to decide whether the positive subgraph is a proper circular arc graph or not, which can be done in $\mathcal{O}(n+m)$ time. Then, if the answer is positive, you have to find an arc model of the positive subgraph with the arcs of the same length. From that arc model,

you define an almost valid drawing and then a valid drawing, which can be done in $\mathcal{O}(n^2)$ time. Hence, decision and construction both together can be done in $\mathcal{O}(n^2)$ time, when the input signed graph is complete.

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